

Heat and Mass transfer

Heat Transfer from an Annular Fin

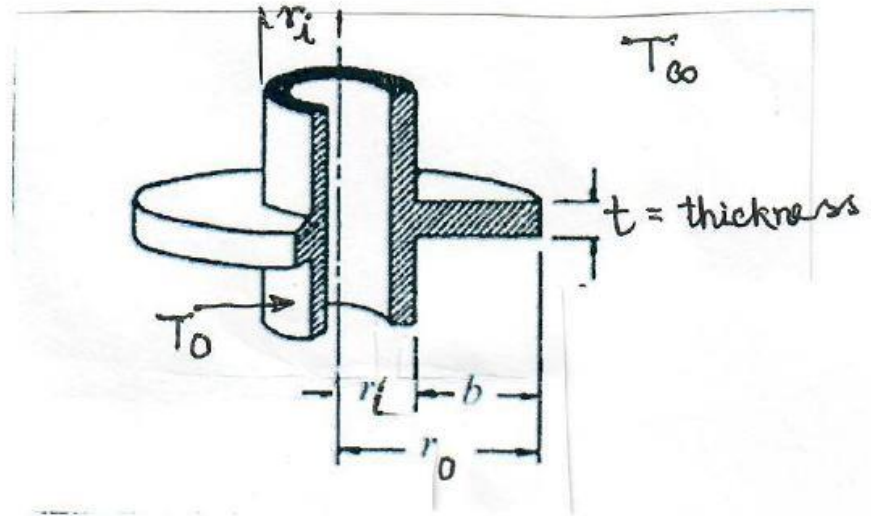
Annular fin with rectangular profile area

- Cross sectional area at any r :

$$A(r) = 2\pi r t$$

$$\frac{dA}{dr} = 2\pi t$$

$$\frac{dA/dr}{A(r)} = \frac{1}{r}$$

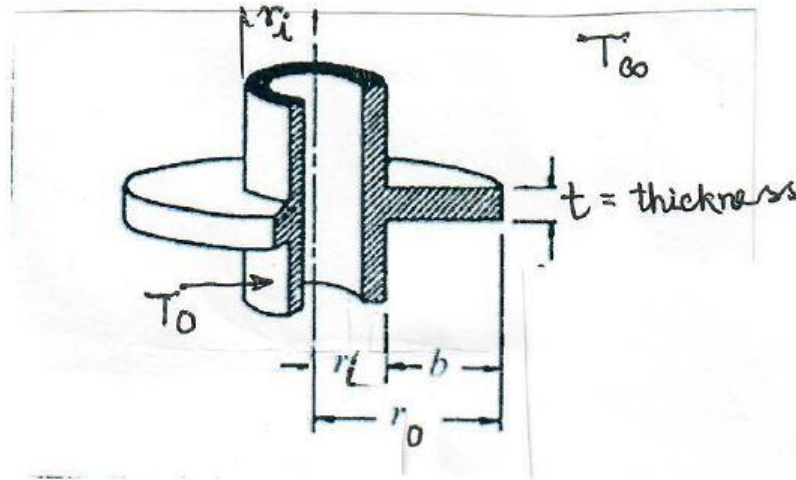


- Surface area at any r : $S(r) = 2\pi (r^2 - r_i^2)$

$$\frac{dS(r)}{dr} = 4\pi r$$

$$\frac{dS/dr}{A(r)} = \frac{2}{t}$$

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➤ Governing Equation:

$$\frac{d^2 T}{dr^2} + \frac{dA/dr}{A} \frac{dT}{dr} - \frac{h(dS/dr)}{kA} (T - T_\infty) = 0$$

$$\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} - \frac{h}{k} \cdot \frac{2}{t} (T - T_\infty) = 0$$

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➤ Substitute $\theta = T - T_{\infty}$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r} \frac{d\theta}{dr} - \frac{2h}{kt} \theta = 0$$

➤ At $r = r_i$, $T = T_0$ or, at $r = r_i$, $\theta = \theta_0$

➤ At $r = r_o$, $-k \frac{dT}{dr} = h(T - T_{\infty})$ or, at $r = r_o$, $-k \frac{d\theta}{dr} = h\theta$

➤ Let $\rho = \beta r^n$: $\frac{d\rho}{dr} = \beta n r^{n-1}$ or, $\frac{d\theta}{dr} = \frac{d\theta}{d\rho} \cdot \frac{d\rho}{dr} = \beta n r^{n-1} \frac{d\theta}{d\rho}$

➤ $\frac{d^2\theta}{dr^2} = \frac{d}{dr} \left[\frac{d\theta}{dr} \right] = \frac{d}{dr} \left[\beta n r^{n-1} \frac{d\theta}{d\rho} \right] = \beta n (n-1) r^{n-2} \frac{d\theta}{d\rho} + \frac{d^2\theta}{d\rho^2} \beta^2 n^2 r^{2n-2}$

So, $\frac{d^2\theta}{d\rho^2} \beta^2 n^2 r^{2n-2} + \frac{d\theta}{d\rho} \beta n (n-1) r^{n-2} + \beta n r^{n-2} \frac{d\theta}{d\rho} - \frac{2h}{kt} \theta = 0$

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$$\text{Or, } \beta^2 n^2 r^{2n-2} \frac{d^2 \theta}{d\rho^2} + \beta n^2 r^{n-2} \frac{d\theta}{d\rho} - \frac{2h}{kt} \theta = 0$$

$$\text{Or, } \beta^2 r^{2n} \frac{d^2 \theta}{d\rho^2} + \beta r^n \frac{d\theta}{d\rho} - \frac{r^2 2h}{n^2 kt} \theta = 0$$

$$\text{Or, } \rho^2 \frac{d^2 \theta}{d\rho^2} + \rho \frac{d\theta}{d\rho} - \frac{r^2 2h}{n^2 kt} \theta = 0$$

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➤ We want the equation to be

$$\frac{d^2\theta}{d\rho^2} + \frac{1}{\rho} \frac{d\theta}{d\rho} - \theta = 0$$

$$\theta = C_1 I_0(\rho) + C_2 K_0(\rho)$$

$$\theta = C_1 I_0(\beta r) + C_2 K_0(\beta r)$$

$$\rho^2 = \frac{r^2}{n^2} \frac{2h}{kt}$$

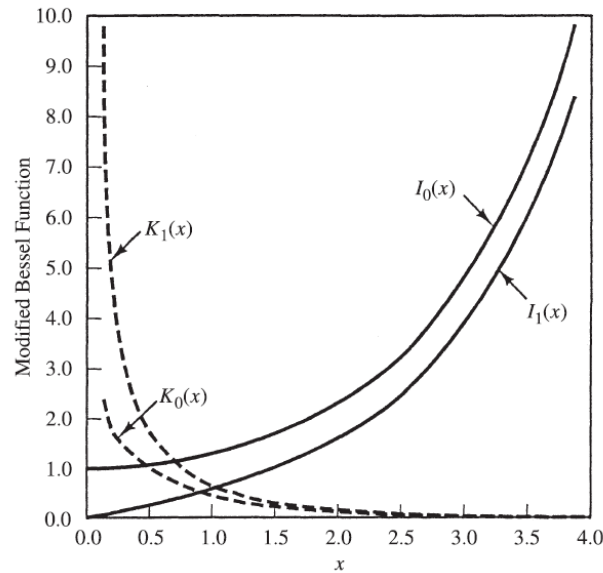
$$\text{or, } \beta^2 r^{2n} = \frac{r^2}{n^2} \frac{2h}{kt}$$

$$n = 1, \quad \beta = \sqrt{\frac{2h}{kt}}$$

$$\rho = \beta r$$

I_0 is modified Bessel function of the first kind (order zero)

K_0 is modified Bessel function of the second kind (order zero)



Recall what we did for the Tapered Fin

$$\theta = C_1 I_0(\beta\sqrt{x}) + C_2 K_0(\beta\sqrt{x})$$

➤ At $x = 0$, $I_0 = 1$, $K_0 \rightarrow \infty \Rightarrow C_2 = 0$

$$\theta = C_1 I_0(\beta\sqrt{x})$$

➤ At $x = L$, $\theta = \theta_0 \Rightarrow \theta_0 = C_1 I_0(\beta\sqrt{L})$ so, $C_1 = \theta_0 / I_0(\beta\sqrt{L})$

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➤ The derived solution is

$$\theta = C_1 I_0(\beta r) + C_2 K_0(\beta r)$$

Two integral formulas for the modified Bessel functions are

$$I_0(\beta r) = \frac{1}{\pi} \int_0^\pi \exp(\beta r \cos u) du,$$

$$I_n(\beta r) = \frac{1}{\pi} \int_0^\pi \cos(nu) \exp(\beta r \cos u) du,$$

$$K_0(\beta r) = \int_0^\infty \exp(-\beta r \cosh u) du$$

and

$$K_n(\beta r) = \int_0^\infty \cosh(nu) \exp(-\beta r \cosh u) du$$

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➤ It is assumed that $\beta r > 0$ everywhere.

$$\text{So,} \quad \frac{dI_0(\beta r)}{dr} = \beta I_1(\beta r) \quad \text{and} \quad \frac{dK_0(\beta r)}{dr} = -\beta K_1(\beta r)$$

Now,

$$\frac{d\theta}{dr} = C_1 \frac{dI_0(\beta r)}{dr} + C_2 \frac{dK_0(\beta r)}{dr} = \beta [C_1 I_1(\beta r) - C_2 K_1(\beta r)]$$

➤ Total heat transfer to the fin from the fin base is

$$(-2\pi r_i tk) \frac{d\theta}{dr} \Big|_{r=r_i} = -2\pi\beta r_i tk [C_1 I_1(\beta r_i) - C_2 K_1(\beta r_i)]$$

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- The functions I_1 and K_1 can be evaluated approximately by using different methods for numerical integration or with the help of MatLab or values can be directly obtained from the tables available in the books:
- Handbook of Mathematical Functions with Formulas, graphs, and Mathematical Tables Edited by Milton Abramowitz and Irene A. Stegun, Dover Publications, 1965
 - A Treatise on the Theory of Bessel Functions by G. N. Watson, Cambridge University Press, 1944.



Thank you